On Reconstructing Dedekind Abstraction Logically

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Abstract: While Richard Dedekind's technical contributions to the foundations of mathematics have been widely praised from the beginning, his remarks about "abstraction" and "free creation", as tied to his structuralist views, have received a more mixed reaction. Often they have been either ignored as irrelevant or dismissed as crudely psychologistic, while occasionally it has been suggested to interpret, or reconstruct, them "logically". But what exactly could the latter amount to? In this essay, four ways of interpreting "Dedekind abstraction" from a logical point of view are explored, called the "neo-Russellian", "neo-Hilbertian", "neo-Fregean", and "neo-Cantorian" reconstruction, respectively. Distinguishing them is meant as a contribution to Dedekind scholarship, but also to current philosophical debates about structuralism.

Richard Dedekind made many technical contributions to the foundations of mathematics, most of which have been widely accepted and praised. Some of his more informal, philosophical remarks in this context have, however, been received more critically. His remarks about "abstraction" and "free creation", as tied to his structuralist views about the natural and real numbers, constitute a main example. They have often been either dismissed as irrelevant or attacked as a crude form of psychologism. In Dedekind's defense, it has been suggested to interpret these remarks not in a problematic psychological but in a "logical

*As my former dissertation advisor and mentor, Bill Tait has had a strong influence on my interests, my research, and my career. I am grateful for his support and inspiration, which continues until today. I am also pleased to be able to contribute a paper on a topic he addressed himself in his writings, even if only tangentially, namely "Dedekind abstraction". Further references will be provided as we go along.

sense", or at least, as having a defensible "logical core", as W. W. Tait put it. But what does that amount to if elaborated more, i.e., how exactly should "Dedekind abstraction" be understood from a logical point of view? This is the core issue to be addressed in the present paper.[1]

The paper is structured as follows. In its first section, some crucial passages from Dedekind's writings, especially from his well-known essay, *Was sind und was sollen die Zahlen?* (1888), will be introduced. In section two, we will turn to several influential criticisms of these passages, by Michael Dummett, Bertrand Russell, and Gottlob Frege, together with some initial, informal defenses of Dedekind against them. In the third section of the paper, four specific and relatively detailed proposals for how to reconstruct Dedekind abstraction logically and more formally will be introduced, called the "neo-Russellian", "neo-Hilbertian", "neo-Fregean", and "neo-Cantorian" reconstructions, respectively, for reasons that will become apparent along the way. This will be followed, in section four, by a further comparative discussion, both about the strengths and weaknesses of these four options and about their Dedekindian credentials. A brief summary and conclusion will round off the paper.

1 Dedekind's Crucial Remarks

There are two passages in Dedekind's *Was sind und was sollen die Zahlen?* that are crucial for present purposes, to be found in Sections 73 and 134 of that text. Before quoting them, let me provide some background. Dedekind's 1888 essay is meant to provide a novel account of the natural numbers $\mathbb{N}$, developed within the framework of a general theory of sets and functions (or "mappings"). Two main ingredients in it are his definitions of what it means for a set to be infinite (Definition 64) and simply infinite (Definitions 71), respectively. As is well known, the latter amounts to a characterization of the natural numbers in terms of (an early version of) the second-order Dedekind-Peano axioms. In between these two definitions, Dedekind presents, as a third main ingredient, a (rather controversial) argument for the existence of an infinite set (Theorem 66).[2] Assuming that result, the existence of a simply infinite subset N then follows straightforwardly (Theorem 72).

The first passage crucial for us occurs as the next step. Dedekind writes:

> 73. Definition. If in the consideration of a simply infinite system N set in order by a mapping Φ we entirely neglect the special character of the elements, simply retaining their distinguishability and

[1]Towards the end of the second section, I will mention an alternative, more "pragmatic" defense of Dedekind's remarks about "abstraction" too. Yet another possible defense, suggested to me by Benis Sinaceur, consists of interpreting Dedekind in a broadly "epistemological" sense. I cannot address that alternative in the present paper, including its relation to a "logical reading", but hope to do so in a future publication.

[2]Dedekind's argument for his Theorem 66 involves an appeal to "the totality S of all things which can be objects of my thought" (Dedekind 1963, p. 64). I will leave aside what is problematic about it here; cf. Reck (2003, 2013) and Klev (2018) for details.

> taking into account only the relations to one another in which they are placed by the order-setting mapping Φ, then these elements are called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series* N. With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind (Dedekind 1863, p. 68; original emphasis, translation modified slightly).

Note here, in particular, Dedekind's use of the terms "abstraction" and "free creation". It is these two terms that call for further clarification (including disambiguation, as our later discussion will make evident).

The second passage crucial for our purposes occurs later in Dedekind's 1888 essay. Here the context is the following: At this point, Dedekind has established that any two simple infinities are isomorphic to each other (Theorem 132), also that any set that can be mapped 1-1 onto a simple infinity is itself simply infinite (Theorem 133). Then he adds:

> 134. Remark. By the two preceding theorems (132), (133) all simply infinite systems form a class in the sense of (34) [an equivalence class]. At the same time, with reference to (71), (73) it is clear that every theorem regarding numbers, i.e., regarding the elements n of the infinite system N set in order by the mapping Φ, and indeed every theorem in which we leave entirely out of consideration the special character of the elements n and discuss only such notions as arise from the arrangement Φ, possesses perfectly general validity for every other simply infinite system Ω set in order by a mapping θ and its elements ν, and that the passage from N to Ω (e.g., also the translation of an arithmetic theorem from one language into another) is effected by the mapping ψ considered in (132), (133), which changes every element n of N into an element ν of Ω, i.e., into $\psi(n)$. [...] By these remarks, as I believe, the definition of the notion of numbers given in (73) is fully justified. [...] (*ibid.*, pp. 95–96, translation modified slightly).

Particularly relevant in this passage is Dedekind's observation (without proof) that any arithmetic theorem about N "possesses perfectly general validity" also for every other simple infinity, via the corresponding isomorphism ψ, and that this fact "fully justifies" his earlier introduction of "the notion of numbers".

A natural and fairly literal (though not uncontroversial) reading of these two passages would seem to be the following: After first establishing some basic theorems about the notion of simple infinity, including its satisfiability (the existence of systems falling under it) and its categoricity (every two systems falling under it are isomorphic), Dedekind introduces, by an act of "free creation", a special such system worthy of the label "the natural numbers".

And he does so by "abstraction" from whatever non-arithmetic properties the elements of the initial simple infinity have, i.e., properties going beyond those definable in terms of the function Φ. Later this procedure is justified further by noting that its result is invariant under the initial choice of a simply infinite system, since all such systems are isomorphic which implies that any arithmetic theorem that holds for one also holds for any other.

According to this reading of Dedekind, the system of "the natural numbers" that is introduced via "abstraction" is different from whatever simple infinity he started with (although it is isomorphic to it). It is a separate, *sui generis* simple infinity. Why and how so? Unlike the elements contained in other simple infinities, its elements are not determined, i.e. characterized in their nature, by non-arithmetic properties; instead, they are determined, purely and fully, by the relevant "structural" properties (cf. Reck 2003). It is this aspect that makes the resulting position a kind of "structuralism", or more specifically, a version of "non-eliminative structuralism".[3] (Again, this reading is not uncontroversial; and it is in need of disambiguation.)

Such an initial reading of Dedekind finds reinforcement if we consider how he proceeds in his other foundational essay, *Stetigkeit und irrationale Zahlen* (1872), with respect to the real numbers $\mathbb{R}$. In that context too, he first introduces a central notion: that of a continuous ordered field. He argues that this notion is satisfiable, i.e., that complete ordered fields exist (the system of all cuts on the rational numbers, endowed with corresponding operations and an ordering, is his main example). And then he introduces "the real numbers" once again by "creation" (Dedekind 1963, p. 15). Admittedly, Dedekind does not use the word of "abstraction" in this earlier essay; nor does he formulate a categoricity theorem for complete ordered fields yet (although it can be added, as is well known). On the other hand, he is more explicit in this earlier text that the mathematical objects introduced along such lines are *sui generis*, since, unlike the elements in other continuous ordered fields, they have no "foreign properties". For example, the real numbers do not have elements in a set-theoretic sense, as Dedekind cuts do.[4] Furthermore, in a letter from 1888—the year in which *Was sind und was sollen die Zahlen?* was published—he makes clear that the two cases are meant to be parallel, including with respect to the issue of "creation" (Dedekind 1888b).[5]

2 Standard Criticisms and Initial Defenses

As mentioned at the beginning of this essay, Dedekind's remarks about "abstraction" and "free creation" were not received as positively as his corresponding technical results. Indeed, an interpretation of Dedekind that was dominant

[3] Cf. Parsons (1990) for this terminology; and see Reck & Price (2000) for more.

[4] On this last point, see also Dedekind (1876), among others.

[5] In an aside in his 1872 essay, Dedekind adds that the negative and fractional numbers should be taken to "have been created by the human mind" as well (Dedekind 1963, p. 4). Hence this amounts to a general theme; cf. Reck (2003) for more.

in English-speaking philosophy for decades, and is still influential today, is to accept our initial reading above while giving it a twist that makes it evidently problematic. A particularly explicit instance of it can be found in Michael Dummett's writings. According to Dummett, Dedekind's position involved the view that "the mind could, by this means [i.e., abstraction], create an object or system of objects lacking the features abstracted from, but not possessing any others in their place" (Dummett 1991, p. 50). Why is this problematic? Because, as Dummett argues, it makes mathematics hopelessly subjective; i.e., its objects exist then only in people's minds or subjective consciousness. This amounts to a crude and problematic form of psychologism.

Dummett is explicit that it is this psychologistic, subjectivist aspect to which he is objecting primarily. Appealing to the authority of Gottlob Frege, whom he ranks far above Dedekind as a philosopher of mathematics, he writes: "Frege devoted a lengthy section of *Grundlagen* [*der Arithmetik*], sections 29–44, to a detailed and conclusive critique of this misbegotten theory" (*ibid.*); and the particular sections of *Grundlagen* invoked contain Frege's criticism of subjectivist views about mathematics, especially views according to which numbers only exist in people's minds. Another authority appealed to in this context is Bertrand Russell. Thus Dummett writes: "[Dedekind] believed that the magical operation of abstraction can provide us with specific objects having only structural properties. Russell did not understand that belief because, very rightly, he had no faith in abstraction thus understood" (p. 52). On the basis of such considerations Dedekind's position is dismissed as "mystical structuralism" by Dummett, while he takes "eliminative" versions of structuralism more seriously, including calling them "hard nosed".

Is Dummett justified in treating Frege and Russell as his allies in this context, and more specifically, in rejecting Dedekind's position as a crude form of psychologism? There is no doubt that these two thinkers are critical of Dedekind's appeal to "abstraction". But on closer inspection this is not (or not primarily) because of its supposed psychologistic character.[6] Russell's main objection is to the alleged structural nature of mathematical objects that he took Dedekind to endorse. As he puts it memorably:

> [I]t is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression. If they are to be anything at all, they must be intrinsically something; they must differ from other entities as points from instants, or colours from sounds. (Russell 1903, p. 249)

Basically, for Russell mathematical objects, such as Dedekind's "ordinals", cannot be characterized purely by their "relational" or "structural" properties; they must have an "intrinsic" nature. More generally, Russell cannot make sense of Dedekind's appeal to "abstraction". At one point, in a seeming attempt to be

[6]For the rest of this section, I will draw on Reck (2013) and Reck (forthcoming a), which contain more detailed discussions of Frege's and Russell's reactions to Dedekind.

charitable, he interprets it as an unclear appeal to his own "principle of abstraction", i.e., the use of equivalence classes for introducing new mathematical objects (more on the latter below).[7]

Frege, in turn, was certainly critical of psychologistic, subjectivist views about mathematics. Yet the discussions in the sections of *Grundlagen* to which Dummett refers (a book published in 1884) and in which Frege discusses views about the nature of the natural numbers do not explicitly mention Dedekind (whose *Was sind und was sollen die Zahlen?* was published only in 1888). In his later *Grundgesetze der Arithmetik, Vol. 2* (1903), Frege does lump Dedekind with a number of writers about mathematics who talk about "creation" in connection with the real numbers; and he does bring up the psychologism charge again all of them. But even in that book, Frege is more charitable to Dedekind than Dummett. If looked at carefully, Frege's primary objection to Dedekind is that he failed to be explicit about the basic laws underlying "abstraction", not that it is problematically psychologistic.[8]

Be that as it may, the following question arises: Do we have to interpret Dedekind's appeal to "abstraction" in a psychologistic sense, whether Frege and Russell accuse him of that or not? In his 1996 paper, "Frege versus Cantor and Dedekind: On the Concept of Number", W. W. Tait takes issue with this interpretation, and especially, with Dummett's dismissal of Dedekind's position. (Similarly for Cantor's appeal to "abstraction", which Dummett dismisses as well.[9]) It is not the structuralist part of Dummett's interpretation of Dedekind that is rejected by Tait, but the psychologistic twist given to it. But what is the alternative? As Tait puts it: "[T]he abstraction in question has a strong claim to the title *logical abstraction*" (*ibid.*, p. 84). He goes on to suggest a particular way of understanding "logical abstraction", namely in terms of how the "sense" of relevant propositions is determined. However, that suggestion is not worked out in technical detail by him.[10]

Tait's suggestion to interpret Dedekind more charitably was picked up in my 2003 article, "Dedekind's Structuralism: An Interpretation and Partial Defense", including an attempt to clarify "Dedekind abstraction" further. In that article, I interpret Dedekind as a "logical structuralist"; yet my interpretation

[7]For an insightful, detailed discussion of Russell's reaction to Dedekind's structuralism and his talk of "abstraction", cf. Heis (forthcoming).

[8]Cf. Reck (forthcoming a); we will come back to this issue briefly in the fourth section.

[9]As Dummett writes: "It was virtually an orthodoxy, subscribed to by many philosophers and mathematicians, including Husserl and Cantor, that the mind could, by this means, create an object or system of objects lacking the features abstracted from, but not possessing any others in their place" (Dummett 1991, p. 50).

[10]For present purposes, the central remark in Tait's article is this: "[W]hat seems to me to be essential to this kind of abstraction is this: the propositions about the abstract objects translate into propositions about the things from which they are abstracted and, in particular, the truth of the former is founded upon the truth of the latter. So the abstraction in question has a strong claim to the title *logical abstraction*: the sense of a proposition about the abstract domain is given in terms of the sense of the corresponding proposition about the (relatively) concrete domain" (Tait 1996, p. 84, emphasis in the original). As we will see below, this suggestion can be elaborated in several different ways.

remained too informal and imprecise in certain respects as well. Apart from that, the suggestion to understand Dedekind in a logical rather than a psychological sense can be traced back further in time, as I found out subsequently.[11] Namely, Ernst Cassirer's 1910 book, *Substanzbegriff und Funktionsbegriff*, anticipates Tait's and my own general defense of Dedekind by almost a century. In it, one can find remarks such as the following:

> [Dedekind's form of abstraction] means logical concentration on the relational system, while rejecting all psychological accompaniments that may force themselves into the subjective stream of consciousness, which form no constitutive moment of this system (Cassirer, 1910, p. 39).

The same point is repeated in various other works by Cassirer, going back to his 1907 article, "Kant und die modern Mathematik", and forward to, e.g., his 1929 book, *The Philosophy of Symbolic Forms, Vol. III*. But while Cassirer was ahead of his time, also in other respects, his discussion of Dedekind abstraction does not include a detailed logical reconstruction of it either. After all, Cassirer was not a mathematical logician.[12]

The main goal of the present article is to supplement Tait's, Cassirer's, and my earlier readings of Dedekind, in the sense of working out a logical reconstruction of "Dedekind abstraction" in some technical detail (or indeed, several such reconstructions). But before turning to that task, let me mention another defense of Dedekind against the psychologism charge, or another way of reading Dedekind's philosophical remarks that is more charitably than Dummett's. The main suggestion in that interpretation consists of rejecting not only the supposed psychologistic side of Dedekind's position, but also the non-eliminative structuralist interpretation sketched above, i.e., the idea that "Dedekind abstraction" introduces a novel system of mathematical objects characterized by their structural properties alone. The resulting interpretation amounts, then, to a form of "eliminative structuralism".

What is the position attributed to Dedekind along such eliminative lines? Let us start again with the case of the natural numbers. (The case of the reals is parallel.) The core idea is this: After Dedekind has constructed a particular simple infinity, what his "abstraction" does is not to introduce a separate, distinguished system of objects, but simply to treat the given system in a novel way, namely by ignoring all its non-arithmetic aspects. And as should be added right away, any other simple infinity would do as well, since they are all isomorphic and the same arithmetic theorems hold of them, as Dedekind observed. What are "the natural numbers", then? Well, pick any

[11] In that respect, I am indebted to Michael Friedman's writings and, especially, to conversations with Pierre Keller. See also Reck (2013) and Yap (2014).

[12] Cassirer works with a broader, less formal notion of "logic" than Frege and Russell, although it is meant to encompass theirs. For more on his reception of Dedekind and the resulting "logical idealism", cf. Reck (forthcoming b) and Reck & Keller (forthcoming).

simple infinity and treat it as "the natural numbers". That is all we need to do for arithmetic purposes, i.e., all that is required for mathematical practice.

In the secondary literature the most subtle, worked-out version of this alternative, presented as a reading of Dedekind, occurs in a new paper by Wilfried Sieg & Rebecca Morris, entitled "Dedekind's Structuralism: Creating Concepts and Deriving Theorems" (this volume). The position attributed to Dedekind is still "structuralist"; but it is "eliminative", insofar as it works without introducing a *sui generis* system of objects besides the initially constructed simple infinity. With respect to the form of structuralism involved, one can say that the approach involves an "indifference to identify", in any absolute sense, the natural numbers with a particular simple infinity, since "any of them will do".[13] The same applies, *mutatis mutandis*, for the reals.

Compared to Dummett's, such a reading of Dedekind certainly has its advantages. However, one can now raise two questions: First, is the resulting position not again problematically psychologistic, since all depends on "ignoring" certain aspects of the initial simple infinity? An immediate response might be this: If we assume that the elements of the initially constructed simply infinity are not "mental objects", we do not end up with "the natural numbers" as such either. Put more positively, the "ignoring" at issue should be understood less in a psychological and more in a "pragmatic" sense. But that leads to a second question: Does this position not also have a "logical core", underneath is "pragmatic" surface; and if so, what is it? In what follows, an answer to the latter question will be suggested as well.[14]

3 Four Logical Reconstructions of Dedekind Abstraction

If one rejects uncharitable criticisms of Dedekind like Dummett's, is not fully satisfied with the "pragmatic" perspective on Dedekind in Sieg & Morris, and is intrigued by Tait's and Cassirer's alternative suggestion, one is left with a task. Namely, how can a "logical" perspective on Dedekind abstraction be spelled out in more detail (including clarifying its relationship to the pragmatic interpretation)? Actually, there is not just one option for a logical reconstruction—I will introduce four possible alternatives in this connection. These will be called the "neo-Russellian", the "neo-Hilbertian", the "neo-Fregean", and the "neo-Cantorian" reconstruction of Dedekind abstraction, respectively, for reasons that will become apparent.

Before going into specifics, let me set up a logical framework for all of these reconstructions. It will not be necessary to adopt this particular framework in the end; but starting with it will contribute to ease of formulation and clarity. (Some alternatives will be mentioned later, e.g., working within higher-order

[13] Compare Burgess (2015), ch. 3. In Reck & Price (2000), this position is called "relativist structuralism"; and a particular, fairly widespread version is "set-theoretic structuralism".

[14] A third question is this: Does such an "eliminative structuralist" reading do justice to Dedekind's repeated remarks about "creation"? This is doubtful, I think, although the interpretive issues are subtle. I plan to address them further in a future publication.

logic supplemented with some existential assumptions.) The kind of framework needed is some general theory of sets and functions, like in Dedekind's original approach. And the particular version of it that I will use, at least provisionally, is ZFC set theory. Actually, at certain points it will be important to talk about proper classes; and we will want to admit urelements too, collected together in a corresponding domain U, besides the pure sets in V. This background theory will allow for the construction of particular systems of objects to which Dedekind abstraction can then be applied.

Within ZFC we can, for example, introduce the finite von Neumann ordinals (starting with 1) to form a particular simple infinite system (with $\{\emptyset\}$ as the initial element, s: $x \to x \cup \{x\}$ as the successor function, and $\omega' = \omega \setminus \{\emptyset\}$ as the domain).[15] Similarly we can introduce a set-theoretic system of cuts on $\mathbb{Q}$ as a particular continuous ordered field. More generally, we can talk about "relational systems" of the form $S = \langle a_1, \ldots, a_n, f_1, \ldots, f_m, R_1, \ldots, R_l, D \rangle$, where the domain D is some set and a_i, f_i, and R_i are defined on it as usual. Each of our four reconstructions of Dedekind will now be characterized in terms of two ingredients: (i) an "abstraction operator", ab, that can be applied to such relational systems S so as to yield "abstract structures"; (ii) a related way of "analyzing" mathematical statements p, e.g., truths of arithmetic such as $2 + 3 = 5$ and $\forall n \forall m(n + m = m + n)$.

3.1 The Neo-Russellian Reconstruction

Above we saw that Russell was critical of what he took to be Dedekind's structuralist conception of mathematical objects, also that he tried to make sense of Dedekind abstraction by assimilating it to his own approach. As he wrote: "What Dedekind intended to indicate was probably a definition by means of the principle of abstraction" (Russell 1903, p. 249). Let us reconsider this suggestion. For relational systems S, let $ec(S)$ be the equivalence class of all sets isomorphic to S within V, i.e., $ec(S) = \{S' : \exists f(f : S' \cong S)\}$. Then Russell's comment leads naturally to the following definition:

(i) $ab_1(S) =_{def} ec(S)$

Here ab_1 is an "abstraction operator" that takes a relational system S as its argument and gives the equivalence class of S under isomorphism as the corresponding value.[16] We can then say that $ab_1(S)$ is "the structure" corresponding

[15] As Dedekind starts the natural numbers with 1 and as this will simplify some details later, I have adjusted the usual treatment in ZFC slightly.

[16] A number of variations are possible here. For example, we can consider not the whole equivalence class (a proper class), but some appropriate set-theoretic part of it; cf. the appeal to "Scott's trick" in the next subsection. We can also replace isomorphism with some weaker equivalence relation, e.g. the kind of "structure equivalence" discussed in Shapiro (1997), pp. 91–93, and then work with the resulting equivalence classes/sets instead. We will come back to some of these variants later.

to S. For instance, $\{\langle a, g, D\rangle : \exists f(f : \langle a, g, D\rangle \cong \langle\{\emptyset\}, s, \omega'\rangle)\}$ is "the structure" corresponding to the finite von Neumann ordinals.

In many mathematical cases, we work with relational systems S that are models of axiomatized theories T_S, i.e., such that $S \models T_S$. If the theory at issue is categorical—like in Dedekind's two central cases: arithmetic (with the second-order Dedekind-Peano axioms) and analysis (with the second-order axioms for complete ordered fields)—it characterizes S completely in a relevant sense. We also have: $ec(S) = \{M : M \models T_S\}$. This provides us with another way to think about "the structure" corresponding to S, namely $ab_1(S) = \{M : M \models T_S\}$; i.e., ab_1 maps S onto the model class of T_S.

As a second main ingredient of the intended position, we want to specify an analysis for every mathematical sentence p, i.e., a way of characterizing what it "really says". Let us start again with the simple case of arithmetic. Let us also assume that we are working with a formal system in which every arithmetic object, function, and relation is defined in terms of the basic constants 1, *suc*, and N, as usual in second-order Peano arithmetic. Given some arithmetic sentence p, we can then consider $p(1, suc, N)$, i.e., the formula in which all defined terms have been replaced by their definitions. If we let ourselves be guided by (i) above, we can re-analyze $p(1, suc, N)$ thus:

(ii) $ab_1(p) =_{def} \forall x \forall f \forall X[PA^2(x, f, X) \rightarrow p(x, f, X)]$

Here PA^2 are the usual second-order Dedekind-Peano axioms and $p(x, f, X)$ is the result of replacing 1, *suc*, and N by x, f, and X, respectively, wherever they occur in p (i.e., these three constants are replaced by corresponding variables of the right type over which we can then quantify). Likewise for analysis and similar theories; i.e., in those cases too we go from p to a corresponding sentence in which only the relevant basic constants, functions, and relations occur, and then we quantify these out in the universalized if-then form indicated.

This way of analyzing arithmetic sentences, or mathematical sentences more generally, was considered seriously by Russell himself, e.g., in his 1901 article "Recent Work in the Philosophy of Mathematics".[17] As he writes:

> Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of anything, then such and such another proposition is true of that thing (Russell 1901, p. 77).

This historical fact justifies the label "neo-Russellian reconstruction" for the current approach further. Note also that (ii) fits together naturally with (i) since we again consider all systems isomorphic to an initially given system S,

[17] Similar remarks can be found in Russell (1903), although in that book his views have already shifted in certain respects and he is not applying this approach to arithmetic any more; see again Heis (forthcoming) for further details.

here by quantifying over all models of T_S. (Or at least, it fits well in the case of a categorical axiom system.)[18]

Considered as a reconstruction of Dedekind, this approach has its main textual support in his remark that every theorem concerning a particular simple infinity "possesses perfectly general validity for every other simply infinite system" (Dedekind 1963, p. 96). The latter is now taken to suggest that it is really a theorem about all simple infinites at the same time, in the logical form made explicit by (ii). Similarly for the real numbers.[19]

3.2 The Neo-Hilbertian Reconstruction

In the neo-Russellian reconstruction of Dedekind abstraction, we talk, in the case of arithmetic, about all simply infinite systems together. An alternative suggestion is to assume that we talk about an arbitrary one. Similarly for other parts of mathematics, including analysis. When this is worked out more formally, it leads to our second reconstruction: the "neo-Hilbertian" one.

How could we implement this second suggestion more formally? Assuming again that we work with ZFC in the background, or with certain familiar extensions of it, one option is to make use of a global choice function for sets or classes. Assume *ch* is such a function, i.e., we pick one of them. Then we can give the following definition, where S is again a set-theoretic relational system:

(i) $ab_2(S) =_{def} \; ch(ec(S))$

In other words, as "the structure" corresponding to S we now let our choice function pick a representative from the equivalence class for it.[20] It is clear that this second form of abstraction is closely related to the first, since we have: $ab_2(S) = ch(ab_1(S))$. Note also the following: If we allow for urelements, then the systems in the equivalence class, thus also in the system picked by *ch*, may contain such urlements. To use the case of arithmetic again for illustration, this means that any of our urelements—say Julius Caesar or some beer mug—may now be, say, the element corresponding to 2 ($= s(\{\emptyset\})$ in $ab_2(\langle\{\emptyset\}, s, \omega'\rangle)$.

This observation reveals the first reason why this second approach deserves the name "neo-Hilbertian reconstruction" of Dedekind abstraction. As Hilbert is reported to have said in the case of an axiomatic approach to geometry: "One must be able to say at all times—instead of points, straight lines, and

[18] It is assumed here that we can characterize the systems S to which we apply abstraction in terms of an axiomatic theory, whether categorical or not. In the non-categorical case, (i) and (ii) can still be formulated but will correspond less closely to each other.

[19] Because of the quantification involved, the resulting position may be called "universalist structuralism"; cf. Reck & Price (2000). A well-known variant of it is Geoffrey Hellman's "modal structuralism" (Hellman 1989). Note that the "non-vacuity problem" that partly motivates Hellman's modal twist does not arise if we work with ZFC in the background.

[20] As hinted at in an earlier footnote, it is possible to avoid working with proper classes here. For example, we can use what is known as "Scott's trick", i.e., work with the set of all systems S' isomorphic to S that are of lowest rank in the ZFC hierarchy, instead of the whole equivalence class. This allows for the use a choice function on sets alone.

planes—tables, chairs, and beer mugs." (That is to say, all our proofs must still go through if we replace geometric objects in a model of Euclidean geometry by other objects, including beer mugs etc., assuming we still deal with a relevant model.) There is also a second reason for appealing to Hilbert in this connection. Instead of using a set-theoretic choice function *ch*, one can work with a Hilbertian ϵ-operator for the same purpose.[21]

What about our second main ingredient in this case, i.e., a corresponding way of analyzing mathematical sentences *p*? This can be taken care of easily by generalizing our abstraction operator ab_2, from relational systems to their ingredients. Consider again the case of arithmetic. Assume we let $ab_2(\{\varnothing\})$ be the element corresponding to $\{\emptyset\}(= 1)$ in $ab_2(\langle\{\emptyset\}, s, \omega'\rangle)$, i.e., the element onto with $\{\emptyset\}$ is mapped by ab_2; similarly for $ab_2(s)$ and $ab_2(\omega')$. Given some arithmetic sentence *p*, we can then stipulate:

(ii) $ab_2(p) =_{def} p(ab_2(\{\emptyset\}), ab_2(s), ab_2(\omega'))$

Here $p(ab_2(\{\emptyset\}), ab_2(s), ab_2(\omega'))$ is the result of replacing '1' by '$ab_2(\{\emptyset\})$', '*suc*' by '$ab_2(s)$', and 'N' by '$ab_2(\omega')$' in $p(1, suc, N)$, parallel to above. What this means is that *p* is mapped onto the corresponding sentence concerning the chosen system $ab_2(\langle\{\emptyset\}, s, \omega'\rangle)$. Similarly for continuous ordered fields etc., i.e., the generalization of this approach should again be clear. (In this case there is no significant dichotomy between the categorical and the non-categorical case, since we work directly with S, or with the equivalence class it induces, not with an axiom system that characterizes S.)

Let me add a few further observations about this neo-Hilbertian reconstruction of Dedekind. As the appeal to an arbitrary representative from the relevant equivalence class indicates, this reconstruction is closely related to the pragmatic interpretation of Dedekind sketched at the end of our second section. In fact, it constitutes a way to bring out the "logical core" of that interpretation.[22] Having said that, there is also a difference. The neo-Hilbertian reconstruction of Dedekind abstraction, as a particular logical reconstruction, is tied to working within a particular formal system, such as ZFC. In contrast, the pragmatic interpretation of Dedekind, as sketched above, is naturally understood to proceed more informally.

Still, the neo-Hilbertian reconstruction clarifies the logic underlying the pragmatic approach to Dedekind, which was left implicit and somewhat ambiguous above. Note, finally, two closely related points: First, the end result is again a form of "eliminative structuralism", since no novel, *sui generis* systems are introduced by ab_2. (And the position is "semi-eliminative" in a more general

[21] Indeed, exactly this kind of approach has recently been explored in the literature on structuralism; cf. Schiemer & Gratzl (2016).

[22] Like the pragmatic position but more explicitly now, the neo-Hilbertian reconstruction of Dedekind amounts to a form of "relativist structuralism", and more specifically, a form of "set-theoretic structuralism"; see again Reck & Price (2000).

sense, in our setup by working with sets.) Second, the neo-Hilbertian reconstruction can point to essentially the same textual evidence as the pragmatic interpretation in terms of being a faithful interpretation of Dedekind.[23]

3.3 The Neo-Fregean Reconstruction

Given some system S, we work with all isomorphic systems in the neo-Russellian reconstruction; and we work with an arbitrary chosen representative in the neo-Hilbertian reconstruction. A variant of the latter, available in some cases, is to use a distinguished system in the equivalence class for S. Perhaps such a system can be picked for strong pragmatic reasons, e.g., when we use the finite von Neumann ordinals as "the natural numbers" since they can be generalized to the transfinite. But such pragmatic reasons are not available in general.

A third alternative is to work with a distinguished system that is new, in the sense that the "abstract" corresponding to S is introduced "purely structurally" (and is not an element of V, thus not of $ec(S)$). The following parallel may motivate such an approach. In current neo-logicism (as initiated by Crispin Wright, Bob Hale, and others), the suggestion is to use "Fregean abstraction principles" for introducing mathematical objects, e.g., "Hume's Principle" for introducing the finite cardinal numbers. Neo-logicists tend to apply these principles to concepts within the context of higher-order logic. But such an approach can be modified and generalized for our purposes, as recent work by Øystein Linnebo and Richard Pettigrew has shown.[24] This leads to our "neo-Fregean reconstruction" of Dedekind abstraction.

Adapting the approach by Linnebo & Pettigrew slightly so that it fits our basic setup, this suggests the following "structuralist abstraction principle":

(i) $ab_3(a, S) = ab_3(a', S') \leftrightarrow \exists f(f : S' \cong S \wedge f(a) = a')$

Here a is meant to be an element of S and a' an element of S'; ab_3 is meant to be a function from V to U; and all elements in U are meant to be introduced via this abstraction principle, while no "non-abstracts" exist in U. Again, the approach deserves to be called "neo-Fregean" since (i') has the form of a neo-Fregean abstraction principle modulo the differences mentioned.

What (i) does is to let an object a, considered relative to some relational system S, correspond to an element $ab_3(a, S)$ in U. As the relation on the right side of the biconditional is clearly an equivalence relation, we could also introduce an equivalence class corresponding to each pair $\langle a, S\rangle$, namely $\{\langle a', S'\rangle : \exists f(f : S' \cong S \wedge f(a) = a'\}$. This would lead us back to the vicinity of Russell. But we do not want to work with such equivalence classes here.

[23] Cf. Sieg & Morris for more on that evidence. A central part of it is Dedekind's remark about "leaving entirely out of consideration the special character of the elements n" in a simply infinite system; cf. his Remark 134 as quoted in Section 1.

[24] Cf. Linnebo & Pettigrew (2014).

Instead, the idea is to work with new and simple objects $ab_3(a, S)$ in U. Moreover, we want to use these new elements to construct a relational system that both "lives entirely in U" and is isomorphic to S. To achieve the latter, we need to define the domain of the resulting system and the relations and functions on it that correspond to those in S.

We can proceed as follows: Given $S = \langle a_1, \ldots, a_n, f_1, \ldots, f_m, R_1, \ldots, R_l, D\rangle$, we let $D' = \{ab_3(a, S) : a \in D\}$ be the new domain. (This modifies our definition above by extending it to $ab_3(D, S)$.) We also "lift" the structural features on S given by a_j, f_j, and R_j and transfer them to D', resulting in a'_j, f'_j, and R'_j, as follows: Let $a'_j = ab_3(a_j, S)$ $(1 \leq j \leq n)$. If f_j is a k-ary function on D $(1 \leq j \leq m)$, $b_1, \ldots, b_k \in D'$, and $c_1, \ldots, c_k \in D$ are such that $ab_3(c_i) = b_i$ $(1 \leq i \leq k)$, we let $f'_j(b_1, \ldots, b_k) = ab_3(f_j(c_1, \ldots, c_k))$. (This extends our definition to $ab_3(f_j, S)$.) And if R_j is a k-ary relation on D $(1 \leq j \leq l), b_1, \ldots, b_k \in D'$, and $c_1, \ldots, c_k \in D$ are such that $ab_3(c_i) = b_i$, we let $R'_j(b_1, \ldots, b_m)$ hold if and only if $R_j(c_1, \ldots, c_m)$. (This extends the original definition further to $ab_3(R_j, S)$.) Finally, we put all of this together: $ab_3(S) =_{def} \langle a'_1, \ldots, a'_n, f'_1, \ldots, f'_m, R'_1, \ldots, R'_l, D'\rangle$. In other words:

(i') $$ab_3(S) =_{def} \langle ab_3(a_1, S), \ldots, ab_3(a_n, S), ab_3(f_1, S), \ldots, ab_3(f_m, S), ab_3(R_1, S), \ldots, ab_3(R_l, S), ab_3(D, S)\rangle$$

I said above that our goal is for $ab_3(S)$ to "live entirely in U" and to be isomorphic to the system S from which it is derived. But there is a problem with the latter in general, as Linnebo & Pettigrew already noted. Namely, if S is non-rigid as a relational system (i.e., allows for non trivial isomorphisms), then the function ab_3 collapses distinct elements of S into the same element in $ab_3(S)$; and this prevents $ab_3(S)$ from being isomorphic to S (often already because of cardinality considerations).[25] In other words, our new abstraction principle (i'), based on (i), does not give us what we want in all cases. Then again, it works as intended for the two systems on which Dedekind focused—the natural numbers and the real numbers—since both of them are rigid.

There are several ways in which one can try to rectify the approach, at least to some degree. One can, for example, introduce additional constants for the crucial elements in S, thereby "rigidifying" the system artificially.[26] However, this has drawbacks when there are many of those crucial elements (perhaps uncountably many); and it is unsatisfactory in other ways as well. As an alternative, one can modify principle (i') by using, not abstraction on single elements a relative to S, but on corresponding sets of elements, thereby using "collective abstraction".[27] But unsatisfactory aspects remain again, which is one reason to consider the more radical forth alternative below.

[25] $\langle 1, +, \times, \mathbb{C} \rangle$ is often given as an example; but a simpler one is the unlabeled graph of two elements with no vertices. Cf. Linnebo & Pettigrew (2014).

[26] Examples are the introduction of the constant 'i' for the imaginary unit in the case of $\mathbb{C}$, and going from an unlabeled to a sufficiently labeled graph.

[27] Cf. Litland (unpublished).

But before moving on to that fourth approach, let me make explicit the other core aspect of the neo-Fregean reconstruction, so as to make our discussion of it parallel to those of the other reconstructions. This second aspect concerns how to analyze mathematical sentences p. Actually, after our treatment of this aspect in the neo-Hilbertian reconstruction it should be clear how to proceed in this case too. It is helpful, once again, to use the example of a sentence p for the natural numbers as a simple illustration:

(ii) $ab_3(p) =_{def} p(ab_3(\{\emptyset\}), ab_3(s), ab_3(\omega'))$

More generally, a mathematical sentence p in the language of a relational system S is analyzed as "the same" sentence for the image of S under ab_3. That is to say, we work with: $ab_3(p) =_{def} p(ab_3(a_1, S), \ldots, ab_3(a_n, S), ab_3(f_1, S), \ldots, ab_3(f_m, S), ab_3(R_1, S), \ldots, ab_3(R_l, S), ab_3(D, S))$.

As in the case of the neo-Russellian reconstruction, we can talk about $ab_3(S)$ as the "abstract structure" that corresponds to S. Note also that $ab_3(S)$ is different from all set-theoretic relational systems because, by construction, it "lives entirely in U". Indeed, since $ab_3(S)$ has been introduced as an abstract, we can say that it is "characterized by its relational or structural properties alone". In any case, ab_3 provides us with a third logical reconstruction of Dedekind's more informal talk about "abstraction". Two final comments: First, the neo-Fregean approach to Dedekind abstraction provides us not only with another reconstruction of Dedekind's remarks about "abstraction", but also of his remarks about "free creation". This corresponds to the fact that what results is a version of "non-eliminative structuralism". It also means, second, that there is additional textual evidence for our third approach to Dedekind, at least if one takes his remarks about "free creation" seriously.[28]

3.4 The Neo-Cantorian Reconstruction

The limitation we encountered for ab_3 in the case of non-rigid systems S may make one wonder if there is not another, less limited approach based on a corresponding "structural abstraction principle". In addition, there is a sense in which the neo-Fregean reconstruction works too much "from the bottom up" from a Dedekindian point of view, which is not entirely satisfactory either.[29] But how else could we proceed? To get more inspiration, let us go back to Dedekind's own remarks.

[28] In Sieg & Morris (2018), the claim is that Dedekind changed his mind in connection with "free creation", and thus, moved from a "non-eliminative" to an "eliminative" form of structuralism in his writings from the 1880s. But as indicated in an earlier footnote, this seems in tension with his repeated talk of "creation" in text from the 1880s.

[29] The neo-Fregean approach starts with individual elements obtained by abstraction and builds a structure out of them, as opposed to working with whole domains or systems of elements from the beginning. This is so even though individual elements a are always considered relative to a system S.

Recall that in the case of a simple infinity Dedekind proposes to "entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting function Φ" (Dedekind 1963, p. 68). At this point, the crucial phrase is: "simply retaining their distinguishability" when introducing new object. What that phrase suggests, on the reading to be pursued now, is to work with a set of "pure units" of the right cardinality, so as then to build "the natural numbers" out of them. Here "pure units" are meant to be mathematical objects only distinguished numerically but not in any other way, i.e., they are "qualitative indiscernibles". Is there a way to reconstruct this informal, intuitive idea more formally and logically?

In working out such an approach further, I will proceed more indirectly than in the previous three cases. Eventually, this will amount to an approach based on another "structural abstraction principle"—parallel to the neo-Fregean reconstruction but not identical with it—used to introduce a new abstraction operator ab_4. To prepare that introduction, I will first introduce two different abstraction operators, to be labeled ab_5 and ab_6, that are more closely related to the neo-Russellian and neo-Hilbertian reconstructions of Dedekind abstraction than to the neo-Fregean one. (In fact, these operators will be defined explicitly by using ab_1 and ab_2.) Eventually the approach will be given a further twist, however, which will lead to ab_4.

Suppose, once again, that we start with a set-theoretic relational system $S = \langle a_1, \ldots, a_n,\ f_1, \ldots, f_m, R_1, \ldots, R_l, D\rangle$, i.e., a relational system consisting of sets constructed in ZFC. In addition, suppose we have a set of urelements D' available that has the same cardinality as D, i.e., so that there exists a bijection g from D to D'. We pick such a set D' and bijection g. We now treat D' as the "abstract" that corresponds to D; i.e., we let $ab_5(D, S) = D'$. We also transfer the structural features from S to D'. This is done parallel to the neo-Fregean approach, but using g instead of ab_3. In other words: We let $a'_j = g(a_j)$, $(1 \leq j \leq n)$. (This defines $ab_5(a_j, S)$.) If f_j is a k-ary function on D $(1 \leq j \leq m)$, $b_1 \ldots, b_k \in D'$, and $c_1, \ldots, c_k \in D$ are such that $g(c_i) = b_i$ $(1 \leq i \leq k)$, we let $f'_j(b_1, \ldots, b_k) = g(f_j(c_1, \ldots, c_k))$. (This defines $ab_5(f_j, S)$.) And if R_j is a k-ary relation on D $(1 \leq j \leq l)$, $b_1, \ldots, b_k \in D'$, and $c_1, \ldots, c_k \in D$ are such that $g(c_i) = b_i$ $(1 \leq j \leq m)$, we let $R'_j(b_1, \ldots, b_m)$ hold if and only if $R_j(c_1, \ldots, c_m)$. (This defines $ab_5(R_j, S)$.) Putting all of this together, we stipulate $ab_5(S) = \langle a'_1, \ldots, a'_n,\ f'_1, \ldots, f'_m,\ R'_1, \ldots, R'_l, D'\rangle$. In other words:

(iii) $$ab_5(S) =_{def} \langle ab_5(a_1, S), \ldots, ab_5(a_n, S), ab_5(f_1, S), \ldots, ab_5(f_m, S), ab_5(R_1, S), \ldots, ab_5(R_l, S), ab_5(D, S)\rangle$$

By proceeding thus, it is clear that $ab_5(S)$ will be isomorphic to S; and this is so even in the non-rigid case, since, as g is a bijection, it does not collapse any elements of S. Basically, we made the two sides isomorphic by construction.

While we thus avoid the main limitation of the neo-Fregean reconstruction, we end up with another problem. To see it, suppose we start with two isomorphic systems S and S'. For our purposes, or those of mathematical structuralism more generally, one would expect that $ab(S) = ab(S')$, i.e., the same "abstract" should correspond to each of them. But this is not guaranteed for ab_5 as just introduced. To be sure, we get: $ab_5(S) \cong ab_5(S')$. (This follows from three facts that are guaranteed, namely: $S \cong S'$; $S \cong ab_5(S)$; and $S' \cong ab_5(S')$.) Yet in the construction of $ab_5(S)$ and $ab_5(S')$ their respective domains may have nothing to do with each other, i.e., they may be disjoint. Having said that, it is at least consistent to assume the following: $(*)$ $ab_5(S) = ab_5(S') \leftrightarrow \exists f(f : S \cong S'))$; or so I want to argue now. I will provide three justifications for that claim, two of them informal and the third more formal. (Each will be informative in its own way.)

Assume, first and most informally, that what U contains are just "pure units", i.e., elements only distinguished numerically but not qualitatively. (Grant me for the moment that this is coherent; it will be justified further below.) Then the domains of our systems $ab_5(S)$ and $ab_5(S')$ consist of such pure units too. But then, all we could ever establish, it would seem, is that the elements of $ab_5(S)$ are distinct from each other, but not that any element of $ab_5(S)$ is distinct from any element of $ab_5(S')$. Second, put aside the idea of "pure units". Instead, assume that U contains simply ordinary urelements, but again enough of them for our purposes. In addition, assume that we do not know anything about these urelements, although they do have distinguishing properties now. Then it seems again to follow that $(*)$ can at least not be proved wrong. Both of these are relatively weak arguments and results, of course.

Next, let us proceed more formally and precisely. Assume still that we are working in ZFC with enough urelements. At this point, assume we have a way to ascertain the specific identities of these urelements, in one way or another, i.e., we know about their specific identities.[30] However, we now replace ab_5 by a closely related operator ab_6, constructed in a two=step process. Given some set-theoretic system S, we first consider $ab_2(S)$, i.e., we map S onto $ch(ec(S))$. Second, we perform the construction just described for ab_5 but now starting with $ab_2(S)$. In other words, we work with:

(iv) $ab_6(S) =_{def} ab_5(ab_2(S))$

Suppose again that we are given two isomorphic systems S and S' as arguments. Then it is clear, by construction, that ab_6 will lead to the same result in both cases, i.e., it will give the same "abstract" as their values.[31]

[30] For this purpose, we can use simple "duplicates" of enough elements of V. There are various ways to explicated this idea, e.g., by adding just one urelement u and then using pairs $\langle a, u \rangle$, for all a in V, to play the role of the needed new (here mixed) elements.

[31] In step one, we will be lead to the same representative $ab_2(S)$, since S and S' determine the same equivalence class, $ec(S)$, on which ch acts; and step two will coincide exactly for both. In other words, from $S \cong S'$ we get $ab_2(S) = ab_2(S')$; and hence, $ab_6(S) = ab_5(ab_2(S))$ $= ab_5(ab_2(S')) = ab_6(S)$.

At this point, is is tempting to work directly with ab_6, i.e., to make it our fourth main abstraction operator. But there are reasons to resist that idea. The main reason is that ab_6 depends too closely on the particular choice of the initial isomorphism g, for each relation system S, to be fully adequate.[32] Instead, this is where we switch gears and use a more "axiomatic" approach, parallel to the neo-Fregean reconstruction. Namely, we introduce an abstraction operator ab_4 that is assumed to satisfy the following abstraction principle:

(i) $[ab_4(S) = ab_4(S') \leftrightarrow \exists f(f : S \cong S')] \wedge [ab_4(S) \cong S]$

As we are now proceeding by means of an "implicit definition", as opposed to defining a_4 explicitly, the crucial question is whether introducing such an abstraction operator is consistent or not. (The same question arises for ab_3, since it was also introduced via an "implicit definition".) The answer is: It is (relatively) consistent, and the construction of ab_6 just considered can serve as a semantic consistency proof for it. (That is why we considered ab_6, which builds on ab_5, in the first place.)

Note the following right away: First, while this approach is similar to the neo-Fregean reconstruction in some respects—especially by working again with a "structuralist abstraction principle"—our new principle (i) does not have the form of a neo-Fregean abstraction principle as usually conceived (i.e., simply involving an equivalence relation on the right-hand side of a biconditional). Instead, it has a more complicated logical form. Second, our new approach provides us directly with an "abstract" for the whole system S, unlike in the neo-Fregean approach. (Note that we do not have to add a separate step (i') here, like earlier.) This corresponds to the fact that we now work "from the top down", not "from the bottom up". And third, the elements of $ab_4(S)$ will be "pure units" again, i.e., they will be "qualitative indiscernibles".

To complete the description of this fourth reconstruction of Dedekind abstraction, a few further remarks are in order. To begin with, what about a corresponding way of analyzing mathematical sentences p? Actually, after our earlier discussion it should be obvious how to come up with such an analysis, namely parallel to the neo-Hilbertian and neo-Fregean reconstructions. In the simple case of arithmetic, this means:

(ii) $ab_4(p) =_{def} p(ab_4(\{\emptyset\}), ab_4(s), ab_4(\omega'))$

Similarly for mathematical sentences p corresponding to other languages and theories, i.e., the approach generalizes just like before.

Next, what justifies calling this approach a "neo-Cantorian reconstruction" of Dedekind abstraction? The reason is this: In his well-known article "Beiträge zur Begründung der transfiniten Mengenlehre I–II" (1895–97), Cantor introduces cardinal numbers corresponding to sets M as follows:

[32]This becomes especially clear when we are dealing with non-rigid systems S, where non-trivial isomorphisms for both S and $ab_6(S)$ exist.

> By the "power" or "cardinal number" of M we mean the general concept, which arises [...] from the set M, in that we abstract from the nature of the particular elements of M and from the order in which they are presented. [...] Since every single element m [of M], if we abstract from its nature, becomes a 'unit', the cardinal number [...] is a definite aggregate composed of units [...] (Cantor 1932, pp. 282–283).

Later in the article, Cantor introduces order types for linearly ordered sets M in the same way:

> By this we understand the general concept which arises from M when we abstract only from the nature of the elements of M, retaining the order of precedence among them. [...] Thus, the order type [...] is itself an ordered set whose elements are pure units [...] (*ibid.*, p. 297).

If we replace the phrase "general concepts" by "relational systems" (including the case of a set with no constants, functions, and relations defined on it), these Cantorian remarks sound like a direct application, or adaptation, of Dedekind abstraction, with the language of "pure units" added.

But is it really such a good idea to appeal to "pure units" in this context? Such attempts have not been viewed positively at least since Frege's criticism of them in his *Die Grundlagen der Arithmetik* (1884). In fact, they have often been dismissed as incoherent. While this is true historically, one can respond as follows: In recent years, the pendulum has started to swing in the other direction, in the sense that several informal defenses and more formal reconstructions of "qualitative indiscernibles" have been proposed, so that they have become more respectable again.[33] My neo-Cantorian reconstruction of Dedekind abstraction constitutes a contributing to that shift. Moreover, admitting such objects seems inevitable if one want to be able to treat the case of non-rigid systems S along non-eliminative structuralist lines.[34]

4 Further Discussion and Comparisons

So far, I sketched four logical reconstructions of Dedekind abstraction, corresponding to the abstraction operators ab_1, ab_2, ab_3, and ab_4. Of these, the

[33]Cf. Assadian (2018) and the references in it.

[34]Besides the defense of Cantor and Dedekind in Tait (1996), cf. the more formal reconstruction of Cantor in Fine (1998), which explicitly involves "pure units". Another related approach is the form of structuralism developed in still unpublished work by Hannes Leitgeb, based on graph theory. Finally, there seems to be a connection to Univalent Foundation. Namely, the Univalence Axiom proposed in the recent literature might be seen as an analogue to, or strong generalization of, principle (i) in the neo-Cantorian approach; cf. Awodey (2014). I am planning to compare my neo-Cantorian reconstruction of Dedekind abstraction more to such approaches in the future, thereby working out its details more fully.

neo-Cantorian approach is the most original, while the others can be found, more or less explicitly, in the literature. I compared all four approaches to some degree already, but a more systematic comparison seems called for. This will concern both historical and systematic aspects.

Starting with the systematic side, let me discuss several general constraints one can adopt in connection with Dedekind's approach, or with structuralist approaches to mathematics more generally. Suppose, once more, that S and S' are relational systems constructed in ZFC, also that we are considering an abstraction operator ab on them. Now consider the following four constraints:

(1) $S \cong S' \leftrightarrow ab(S) = ab(S')$

(2) $S \cong ab(S)$

(3) $S' \neq ab(S)$ for all S' in V

(4) $ab(S)$ is characterized by its structural features alone.

How do our four reconstructions fare with respect to these constraints?

It is clear that ab_1 satisfies condition (1), since isomorphic systems determine the same equivalence class. If we work with a fixed choice function ch, as intended above, then (1) is also satisfied by ab_2, as is not hard to see. (1) is true for the operator ab_3 as well; indeed, this is ensured by construction. Concerning our fourth operator and approach, the situation was more interesting. Given the way in which we proceeded initially, via ab_5, there was no guarantee that (1) would hold. But I argued that for the modified operator ab_6 condition (1) is at least consistent. And in the end, we worked with ab_4, as introduced by the corresponding abstraction principle (i), which ensures condition (1) directly. In contrast, for the other approaches (1) is a more indirect, derived feature. That fact, in itself, reveals a main difference between ab_4 and the other three abstraction operators.

With respect to condition (2), the situation looks different in several respects. (2) is false for ab_1, since a set-theoretic system S and the corresponding equivalence class $ec(S)$ are clearly not isomorphic. On the other hand, (2) is true (in full generality) for ab_2, by construction. For ab_3, (2) holds for rigid systems S, while it fails for the non-rigid cases, as Linnebo & Pettigrew pointed out. In contrast, (2) is true for ab_4 (in full generality), once more by construction, assuming it is consistent. Note also that there is a close parallel between ab_2 and ab_4, as the argument for the (relative) consistency of (1) above indicates. On the other hand, ab_2 and ab_4 are quite different in other ways, especially insofar ab_2, like ab_1, is introduced by an explicit definition, while ab_4, like ab_3, is defined "implicitly" or "axiomatically".

Next, let us consider conditions (3) and (4) together. What (3) says is that $ab(S)$ is different from all set-theoretic relational systems. This is true for ab_1, because sets and proper classes are different.[35] In contrast, (3) is false

[35]If we use "Scott's trick" to replace the relevant proper classes by sets, this changes.

for ab_2, since $ab_2(S)$ is a set-theoretic relational system, even thought we don't know which one. For ab_3 and ab_4 condition (3) is true by construction, since we introduced them as functions from SV to U. Moreover, for both ab_3 and ab_4 condition (4) is true, because in each case abS is characterized completely by a structural abstraction principles (in two different ways). In contrast, for ab_1 and ab_2 condition (4) is false, since here $ab(S)$ is a set- or class-theoretic objects that has additional, non-structural characteristics. Finally, both ab_3 and ab_4 lead to "non-eliminative" structuralist positions, while ab_1 and ab_2 lead to versions of "eliminative" structuralism.

Which of our four reconstructions is grounded most firmly in Dedekind's texts, i.e., which of them is most defensible as an interpretation of Dedekind? I already indicated that each has some textual support. Thus, the neo-Russellian approach picks up on the observation, in Dedekind's Remark 134, that "every theorem in which we leave entirely out of consideration the special character of the elements n and discuss only such notions as arise from the arrangement Φ, possesses perfectly general validity for every other simply infinite system Ω". It interprets Dedekind as saying that every arithmetic theorem reveals itself, if analyzed, as a theorem about all simple infinities. Similarly, this approach can make good sense of Dedekind's suggestion to "neglect the special character of the elements" in a given simple infinity, namely by generalizing over all of them. The neo-Hilbertian reconstruction makes sense of the latter remark somewhat differently, namely by replacing an initially constructed simple infinity by an arbitrarily chosen one, where we don't know anything about the "special character of the elements", except that we are still dealing with set-theoretic objects.

Both along neo-Fregean and neo-Cantorian lines, Dedekind's remarks about "free creation", in connection with his remarks about "abstraction", are taken much more seriously than along neo-Russellian and neo-Hilbertian lines. However, doing so does not commit one to psychologism. Instead, what results in each case is a logical reconstruction, via a respective abstraction operator and corresponding structural abstraction principle. From a neo-Fregean and a neo-Cantorian perspective, what Dedekind meant to "create" was distinguished, *sui generis* "abstracts", different from the initial set-theoretic systems. For each of them, this takes the form of working with urelements, or with systems of such urelements, introduced via the structural abstraction principles. Consequently, we end up with forms of "non-eliminative structuralism", which satisfy constraints (3) and (4), in addition to (1) and (2).

Given all of this, how should Dedekind be interpreted overall? Interpretive charity would seem to require not to read him in an problematic psychologicist way, at least if there are alternatives. Whether to interpret him more along neo-Fregean or neo-Cantorian than neo-Russellin or neo-Hilbertian lines depends on how seriously we take his talk about "creation". As he only makes a few (pregnant but ambiguous) remarks in this connection, it is hard to be sure. It is also possible that Dedekind changed his relevant views over time. Whether that is the case is a subtle matter, I believe, one I do not intend to decide conclusively in the present paper.[36] Having said that, I hope that the main interpretive choices available in this context have become clearer, namely by revealing their underlying "logic".

Supposed one is inclined to interpret Dedekind as a non-eliminative structuralist. Is the neo-Fregean or the neo-Cantorian reconstruction closer to his texts? Three arguments speak in favor of the neo-Cantorian option, I believe. First, Dedekind's approach seems "top down", corresponding to the neo-Cantorian reconstruction, than "buttom up", as represented by the neo-Fregean reconstruction. Second, we already noted Dedekind's remark about "neglecting the special character of the elements" in a simple infinity while "retaining their distinguishability", which seems to point to the idea of "pure units". Third, there is the historical link between Dedekind and Cantor, including the fact that they may have influenced each other. Actually, the precise relationship between Dedekind and Cantor concerning this point is a question that seems worth more historical research; and that may lead to further insights into how to interpret either one of them.[37]

Unlike Cantor, Frege was quite skeptical about "pure units". How might Frege have reacted to a neo-Fregean reconstruction of Dedekind, though? This is a very speculative question. But there is one striking detail in Frege's *Grundgesetze der Arithmetik, Vol. II* that may be worth mentioning. After having criticized a number of views about "creation" in mathematics vehemently, in Section III of that book, Frege contrasts them with his own approach in which classes are introduced as extensions of concepts (or as value ranges of functions). Now, his crucial Basic Law V has the form of a neo-Fregean abstraction principle. Also, at this exact point in the discussion he asks the following question: "Can our procedure be called a creation?" (Frege 1903, p. 149) And quite surprisingly, he does not reject this view outright; instead he responds: "The discussion of this question can easily degenerate into a verbal quarrel. In any case, our creation, if one wishes so to call it, is not unconstrained and arbitrary, but rather the way of proceeding, and its permissibility, is settled once and for all (*ibid.*)". I take Frege's main point to be that a systematic way of introducing mathematical objects is called for, and specifically, one working with explicit basic laws. But if so, he may have found our neo-Fregean reconstruction of Dedekind not only worth investigating but even congenial.

Frege did not, of course, work within a set-theoretic framework like ZFC, but in higher-order logic supplemented by Basic Law V. This brings us back to the choice of a general framework. My discussion in this paper was framed in terms of ZFC (plus proper classes and urlements). But this was mostly for ease of presentation and because of its relative familiarity. Indeed, higher-

[36]In Reck (2003), I argued that Dedekind's remarks about "creation", as closely tied to "abstraction", should be taken seriously. In Sieg & Morris (2018), a careful case is made that Dedekind changed his corresponding views over time.

[37]Tait (1996) can serve as the starting point for such an investigation. A related question is whether my neo-Cantorian reconstruction of Dedekind is closest to the interpretation of both Cantor and Dedekind given in Tait (1996). I assume so, but am not altogether sure.

order logic provides an almost as familiar and general background theory for sets/classes and functions too, at least if supplemented with existence principles for them. This suggests that one could present each of my four approaches within such a framework as well, and especially, the neo-Fregean and neo-Cantorian reconstructions of Dedekind abstraction.[38] In fact, what I meant to provide in this paper was four general recipes for how to implement approaches compatible with Dedekindian remarks, given some suitable formal framework.

5 Summary and Conclusion

Let me summarize the discussion in this paper briefly. We started with Richard Dedekind's remarks about "abstraction" and "free creation" in his essay *Was sind und was sollen die Zahlen?* and related writings. While these remarks are often dismissed as a problematic form of psychologism, the paper picked up on a suggestion by W. W. Tait, earlier also by Ernst Cassirer, to interpret them in a "logical" way instead. This is not the only possible defense of Dedekind against the psychologism charge, as a brief interlude about a more pragmatic reading of Dedekind indicated. But even with respect to that reading, the question of its "logical core" arises.

As the paper then illustrated, there are four different ways to reconstruct Dedekind abstraction "logically". For reasons provided along the way, these were called the neo-Russellian, the neo-Hilbertian, the neo-Fregean, and the neo-Cantorian reconstruction, respectively. Each of them was specified in terms of two ingredients: (i) the logical form Dedekind abstraction on relational systems S takes, as spelled out in terms of a corresponding abstraction operator ab; (ii) the way in which mathematical formulas p are re-analyzed accordingly. The resulting positions were compared further, both with respect to the forms of structuralism they embody and their Dedekindian credentials.

What was my basic goal in providing all these reconstructions of Dedekind? The present paper is part of a bigger effort of providing a philosophical interpretation of Dedekind's works. There is thus a more general exegetic project in the background. Yet it seems to me that, before one can argue conclusively for a particular interpretation of any thinker, one should clarify what the main options are. It also helps to work out these options in formal detail, since certain ambiguities or fine distinctions become evident only that way. The most basic goal of this paper was, then, to do the preliminary work of exploring the space of alternatives for interpreting Dedekind abstraction.

Beyond questions of exegesis, one may wonder how Dedekind's remarks, or approaches inspired by them, fit into contemporary debates about structuralism. As we saw, several Dedekindian forms of structuralism are possible, with

[38] Another option would be to work within constructive type theory, as introduced by Per Martin-Löf. In that context, one may again see Univalent Foundations as a natural development of my neo-Cantorian approach to Dedekind. It might even be possible to trace a historical line from Dedekind through category theory to UF.

different strengths and weaknesses. By "Dedekindian forms of structuralism" I mean positions that work with the kind of abstraction operators our four reconstructions illustrate. Seen from that perspective, the basic outcome of the present paper is that Dedekind's talk of "abstraction" can indeed be reconstructed "logically", as W. W. Tait suggested, and that doing so reveals its continuing systematic relevance.[39]

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[39]This paper has been in the works for a while. Earlier versions were presented in a number of contexts, including: the Munich Center for Mathematical Philosophy, October 2012; the Montréal Inter-University Workshop on the History and Philosophy of Mathematics, November 2012; the *Frege-Dedekind Fest*, University of California at Irvine, April 2016; and the conference *Varieties of Mathematical Abstraction*, University of Vienna, August 2018. I am grateful for the comments I received at these events, as well as for the invitations to participate in them in the first place. I am especially indebted to Øystein Linnebo and Georg Schiemer, both for their own work on this topic and for their constructive comments on my approach. The remaining problems should be attributed entirely to me, of course.

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